An Elementary Proof that Schoenberg's Space-Filling Curve Is Nowhere Differentiable

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Up to the end of the nineteenth century (and in six, randomly selected, contemporary calculus books) a plane curve was (and is) defined as the "graph of a pair of parametric equations

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\} t \in I \tag{1}$$

in which the functions f, g are continuous on the interval I", or by words to that effect. This "conventional wisdom" was shattered in 1890 when G. Peano **[1]** demonstrated that this definition embraces what are now called "space-filling curves," that is, curves that pass through every point of a two-dimensional region with positive content such as a square. Since one does not ordinarily call a square a curve, one has to place restrictions on f and g for (1) to produce what one might conventionally call a curve. For example, if one assumes that the mapping $(f, g): I \to E^2$ is continuous and injective, one obtains what is generally referred to as a *Jordan arc* that is, by a theorem of Netto ("a bi-jective map from a line onto a surface is, by necessity, discontinuous" **[2]**) not space filling.

Peano's pioneering work spawned numerous other examples of space-filling curves, one of which, namely Schoenberg's, will be the object of this discussion.

To produce Schoenberg's curve, let $I = [0, 1]$ and let f, g in (1) be defined by

$$f(t) = \frac{1}{2} \sum_{k=0}^{\infty} p(3^{2k}t)/2^k, \qquad g(t) = \frac{1}{2} \sum_{k=0}^{\infty} p(3^{2k+1}t)/2^k \tag{2}$$

where the 2-periodic even function p is defined as in FIGURE 1. (See also **[3]**.)

It is not difficult to demonstrate that f, g are continuous on [0, 1] and that (1), with f, g defined as in (2), represents a space-filling curve (see **[3]** or **[4**, p. 365] or **[5**, p. 438]).

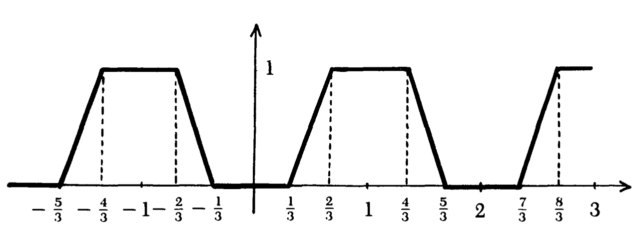

FIGURE 1

Some space-filling curves (but not all) are nowhere differentiable and Schoenberg's is one of them. This was not proved until 1981 when J. Alsina published a fairly complicated and laborious proof **[6]**. (By contrast, Lebesgue's space-filling curve is a.e. differentiable **[4**, p. 365].) Schoenberg himself proved in **[7]** that for $0 < a < 1$, b odd, and $ab > 4$, the function

$$\sum_{k=0}^{\infty} a^k E(b^k t)$$

is nowhere differentiable, where E is linear between consecutive integers and $E(n) = (-1)^n$ for all integers n. He did this by a clever adaptation of Rudin's proof of the nondifferentiability of Weierstrass' first example of such a function ([8, pp. 125–127]). He used this result, in turn, to establish the nowhere differentiability of his curve.

Our proof that Schoenberg's curve is nowhere differentiable is straightforward and does not utilize any advanced notions and techniques. It represents a suitable modification of a proof that the function

$$\sum_{k=1}^{\infty} s_k(t)$$

with $s_1, s_2, s_3, \ldots$ defined as in FIGURE 2, is nowhere differentiable ([9]) and is based on the following.

LEMMA. *If $f:[0,1] \to R$ is differentiable at $t \in (0,1)$, then, for any two sequences $\{a_n\} \to t$, $\{b_n\} \to t$ with $0 < a_n < t < b_n < 1$, by necessity,*

$$\lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n} = f'(t) \tag{3}$$

exists.

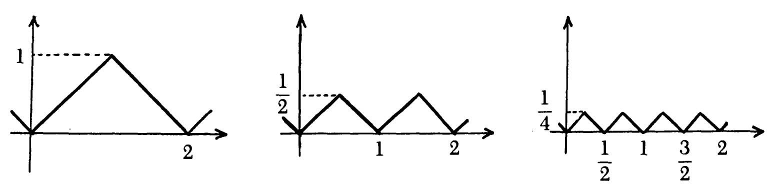

FIGURE 2

The proof of this lemma consists of the observation that

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} - f'(t) = \frac{b_n - t}{b_n - a_n}\left(\frac{f(b_n) - f(t)}{b_n - t} - f'(t)\right)$$
$$+\left(1 - \frac{b_n - t}{b_n - a_n}\right)\left(\frac{f(a_n) - f(t)}{a_n - t} - f'(t)\right)$$

and that

$$\left|\frac{b_n - t}{b_n - a_n}\right|, \left|1 - \frac{b_n - t}{b_n - a_n}\right|$$

are bounded.

We are now ready to establish our main result:

THEOREM. *The functions f, g as defined in* (2) *are nowhere differentiable.*

Proof. (i) First, let $t = 0$. Choose $b_n = 1/9^n$ and consider

$$f(0) = 0$$
$$f(1/9^n) = \frac{1}{2}\sum_{k=0}^{\infty} p(9^k/9^n)/2^k.$$

Since

$$p(9^k/9^n) = \begin{cases} 0 & \text{for } k < n \\ 1 & \text{for } k \geqslant n, \end{cases}$$

we have

$$f(1/9^n) = \frac{1}{2}\sum_{k=n}^{\infty} 1/2^k = 1/2^n,$$

and hence

$$\frac{f(b_n) - f(0)}{b_n} = \left(\frac{9}{2}\right)^n \to \infty$$

as $n \to \infty$, i.e., $f'(0)$ does not exist.

(ii) Next, let $t = 1$ and choose $a_n = 1 - 1/9^n$ to obtain

$$\frac{f(1) - f(a_n)}{1 - a_n} = 9^n - \left(\frac{9}{2}\right)^n \to \infty$$

as $n \to \infty$, i.e., $f'(1)$ does not exist.

Observe, that we did not need the lemma for these two cases but only the definition of the derivative as the limit of an average rate of change.

(iii) Finally, let $t \in (0, 1)$. To achieve our objective, we have to find for every such t two sequences $\{a_n\}, \{b_n\}$ that satisfy the requirements of the lemma so that the limit in (3) does not exist. We will do this for the function f.

Let

$$k_n = [9^n t], \tag{4}$$

where $[x]$ denotes the largest integer that is less than or equal to x, and let

$$a_n = k_n/9^n, \qquad b_n = k_n/9^n + 1/9^n.$$

It is a simple matter to show that, for sufficiently large n,

$$0 < a_n < t < b_n < 1$$

and that $\{a_n\} \to t$, $\{b_n\} \to t$.

From (4), infinitely many k_n are even or infinitely many k_n are odd, or both. We will assume for the sequel that infinitely many of them are *even* and denote the corresponding subsequence again by k_n in order to avoid double subscripts.

From (2) and (4),

$$f(a_n) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p\left(\frac{9^k}{9^n} k_n\right), \qquad f(b_n) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p\left(\frac{9^k}{9^n} k_n + \frac{9^k}{9^n}\right).$$

Hence,

$$\begin{aligned} f(b_n) - f(a_n) &= \frac{1}{2}\sum_{k=0}^{n-1} \frac{1}{2^k}\left[p\left(\frac{9^k}{9^n} k_n + \frac{9^k}{9^n}\right) - p\left(\frac{9^k}{9^n} k_n\right)\right] \\ &\quad + \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}\left[p\left(\frac{9^k}{9^n} k_n + \frac{9^k}{9^n}\right) - p\left(\frac{9^k}{9^n} k_n\right)\right] \triangleq S_1 + S_2. \end{aligned}$$

If $k<n$, then $9^k/9^n \leqslant 1/9$ and, in order to obtain a lower estimate for S_1, we assume the worst possible situation where $(9^k/9^n)k_n+(9^k/9^n)$, as well as $(9^k/9^n)k_n$ both lie in an interval where p descends with slope -3 (see FIGURE 1). Then,

$$p\left(\frac{9^k}{9^n}k_n+\frac{9^k}{9^n}\right)-p\left(\frac{9^k}{9^n}k_n\right)\geqslant -3(9^k/9^n).$$

Hence,

$$S_1\geqslant -\frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}3\frac{9^k}{9^n}=-\frac{3}{9^n\cdot 2}\sum_{k=0}^{n-1}(9/2)^k=-\frac{3}{7\cdot 9^n}\left[(9/2)^n-1\right]. \tag{5}$$

If $k\geqslant n$, then $(9^k/9^n)\geqslant 1$, odd. Hence, $(9^k/9^n)k_n$ is even and $(9^k/9^n)k_n+(9^k/9^n)=\text{even}+\text{odd}=\text{odd}$. Therefore,

$$S_2=\frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}[p(\text{odd})-p(\text{even})]=\frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}=1/2^n. \tag{6}$$

From (5) and (6),

$$\frac{f(b_n)-f(a_n)}{b_n-a_n}=9^n(S_1+S_2)\geqslant\frac{4}{7}\left(\frac{9}{2}\right)^n+\frac{3}{7}\to\infty$$

as $n\to\infty$, i.e., $f'(t)$ does not exist.

We assumed for the preceding argument that infinitely many of the k_n are even. If infinitely many of them are odd, we reverse our strategy and seek an upper bound for the difference quotient. We obtain by analogous reasoning that

$$\frac{f(b_n)-f(a_n)}{b_n-a_n}\leqslant-\frac{4}{7}\left(\frac{9}{2}\right)^n-\frac{3}{7}\to-\infty.$$

Since $g(t)=f(3t)$ by (2), g is also nowhere differentiable.

That the Schoenberg curve is nowhere differentiable as opposed to the Lebesgue curve that is differentiable almost everywhere finds a very graphic expression in the behavior of their approximating polygons [4, FIGURES 2 and 8].

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